Conductivity Due to Classical Phase Fluctuations in a Model For High-T$_c$ Superconductors

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We consider the real part of the conductivity, $\sigma_{1;\alpha\alpha}(\omega)$, arising from classical phase fluctuations in a model for high-T$_c$ superconductors. We show that $\int_0^\infty \sigma_{1;\alpha\alpha}d\omega \neq 0$ below the superconducting transition temperature T_c, provided there is some quenched disorder in the system. Furthermore, for a fixed amount of quenched disorder, this integral at low temperatures is proportional to the zero-temperature superfluid density, in agreement with experiment. We calculate $\sigma_{1;\alpha\alpha}(\omega)$ explicitly for a model of overdamped phase fluctuations.

Because of their high transition temperatures, small coherence lengths, and low superfluid densities, the cuprate superconductors are strikingly influenced by phase fluctuations of the superconducting order parameter. Such fluctuations are largely responsible for flux lattice melting[1] and vortex glass[2] transitions in a finite magnetic field. In addition, they strongly affect the zero-field transition[3], and possibly also the superconducting transition temperature itself in underdoped materials[4]. Phase fluctuations also influence the transport properties of the high-T$_c$ materials. For example, the finite-frequency conductivity shows a fluctuation-induced peak near T_c[5].

Recent measurements in the most anisotropic high-T$_c$ materials, such as Bi$_2$Sr$_2$Ca$_2$Cu$_2$O$_{8+\delta}$, (BSCCO) have found that the real part of the conductivity, $\sigma_1(\omega)$, not only has a peak near T_c but also remains quite large even far below T_c[6]. Published measurements are available typically at frequencies in the 30-200 GHz range[6,7]. In some cases, $\sigma_1(\omega, T)$ at such low temperatures exceeds the peak values observed near T_c[8]. These large values occur over a broad concentration range varying from overdoped to optimally doped to underdoped. Finally, large low-temperature values of $\sigma_1(\omega, T)$ are correlated with a relatively large in-plane superfluid density $n_s(0)$ at $T = 0$.

In this paper, we show that if the phases are assumed to fluctuate *classically*, then the presence of *quenched disorder* will produce a low-temperature background in $\sigma_1(\omega)$ which has many similarities to that observed in experiment[8]. Specifically, we argue that in a sample with quenched disorder there must inevitably be a low-temperature background $\sigma_1(\omega, T)$ arising from classical phase fluctuations, whose frequency integral is related to the zero-temperature superfluid density, $n_s(0)$. While the exact frequency dependence of this background depends on the particular dynamics obeyed by the phase fluctuations, the frequency integral is independent of the dynamics, but depends only on the assumption that the phases fluctuate *classically*. These fluctuations thus provide an alternative possible source of conductivity background, in addition to the gapless quasiparticles which should exist in a d-wave superconductor[9].

In order to model the phase fluctuations, we adopt a *classical XY model* on a tetragonal lattice with different couplings in the ab and c directions[10]. We consider a lattice model of a superconductor, such that each lattice point i is characterized by a phase θ_i. In the presence of a vector potential, the interaction Hamiltonian for this system is given by

$$H = -\sum_{\langle ij;\parallel \rangle} J_{ij;\parallel} \cos(\theta_i - \theta_j + A_{ij})$$
$$ -\sum_{\langle k\ell;\perp \rangle} J_{k\ell;\perp} \cos(\theta_k - \theta_\ell + A_{k\ell}), \qquad (1)$$

where the first sum runs over nearest neighbors in the ab plane and the second over bonds in the c direction; $J_{ij\parallel}$ and $J_{ij;\perp}$ are the couplings between lattice points in the ab and c directions. The gauge factor is $A_{ij} = (2\pi/\Phi_0)\int_i^j \mathbf{A}\cdot \mathbf{d\ell}$, where $\mathbf{A}$ is the vector potential, $\Phi_0 = hc/q$ is the flux quantum, and q is the magnitude of the charge of a Cooper pair.

The diagonal components $n_{s;\alpha\alpha}$ of the superfluid density tensor are given by $n_{s;\alpha\alpha} = [(m^* c^2/(Vq^2)](\partial^2 F/\partial A_\alpha^2)_{A_\alpha=0}$. Here m^* is the mass of a Cooper pair, F the Helmholtz free energy, V the sample volume, c the speed of light, and A_α is a fictitious uniform vector potential applied in the α direction in the presence of periodic boundary conditions in all three directions. Using $F = -k_B T \ln \Pi_{i=1}^N \int_0^{2\pi} d\theta_i \exp(-H/k_B T)$, where N is the number of lattice points, we obtain

$$n_{s;\alpha\alpha} = \frac{m^* c^2}{Vq^2}(\gamma_{1;\alpha} - \gamma_{2;\alpha}). \qquad (2)$$

Here $\gamma_{1;\alpha} = [(2\pi/\Phi_0)a_\alpha]^2 E_\alpha$, where a_α is the lattice constant in direction α ($a_x = a_y \neq a_z$ for the tetragonal lattice), $\Phi_0 = hc/q$ is the flux quantum, and $E_\alpha = \langle \sum_{\langle ij\rangle\parallel\alpha} J_{ij}\cos\theta_{ij}\rangle$, the sum running over bonds in

the direction α. Similarly, $\gamma_{2;\alpha} = [V^2/(c^2 k_B T)]\langle \mathcal{J}_\alpha(t)^2 \rangle$, where $\mathcal{J}_\alpha(t) = (a_\alpha/V)\sum_{\langle ij \rangle \| \alpha}(qJ_{ij}/\hbar)\sin\theta_{ij}$, $\theta_{ij} = \theta_i - \theta_j$, the sums are carried out over distinct bonds in the α direction, and the triangular brackets denote a canonical thermal average.

The key point is to note that $\mathcal{J}_\alpha(t)$ is the volume-averaged supercurrent density in direction α. From this connection, we can use the Kubo formula[11], in the classical limit, together with eq. (2), to obtain an expression for $\sigma_{1;\alpha\alpha}(\omega)$, the real part of the long-wavelength conductivity in the direction α:

$$\sigma_{1;\alpha\alpha}(\omega) = \frac{V}{k_B T}\int_0^\infty dt\,\cos(\omega t)\langle \mathcal{J}_\alpha(t)\mathcal{J}_\alpha(0)\rangle$$
$$= \frac{a_\alpha^2}{V k_B T}\int_0^\infty dt\,\cos(\omega t)\langle \mathcal{I}_\alpha(0)\mathcal{I}_\alpha(t)\rangle, \qquad (3)$$

where $I_{c;ij} = qJ_{ij}/\hbar$ and

$$\mathcal{I}_\alpha(t) = \sum_{\langle ij \rangle \| \alpha} I_{c;ij}\sin\theta_{ij}(t). \qquad (4)$$

Integrating eq. (3) over frequency leads to $\int_0^\infty \sigma_{1;\alpha\alpha}(\omega)d\omega = [V\pi/(2k_B T)]\langle \mathcal{J}_\alpha^2 \rangle$. Substituting this expression into eq. (2), and using $\sigma_{\alpha\alpha}(-\omega) = \sigma_{\alpha\alpha}(\omega)$, we finally obtain

$$S_{1;\alpha} \equiv \int_0^\infty \sigma_{1;\alpha\alpha}(\omega)d\omega$$
$$= \frac{\pi c^2}{2}\left[\left(\frac{2\pi a_\alpha}{\Phi_0}\right)^2 \frac{E_\alpha}{V} - \frac{n_{s;\alpha\alpha}q^2}{m^* c^2}\right]. \qquad (5)$$

We first show that, for an *ordered* system, $S_{1;\alpha}^o(T) = 0$ at $T = 0$. Suppose that $J_{ij} = J_\alpha$ for all bonds in the α direction. Then the phases θ_i are all equal and, with periodic boundary conditions, $E_\alpha = NJ_\alpha$. Similarly, in the limit $T \to 0$, $(\partial^2 F/\partial A_\alpha^2)_{A_\alpha = 0} = (\partial^2 H/\partial A_\alpha^2)_{A_\alpha = 0} = (2\pi a_\alpha/\Phi_0)^2 NJ_\alpha$, from direct evaluation of the second derivative, with periodic boundary conditions. Hence, both terms on the right-hand side of eq. (5) are equal, and $S_{1;\alpha}^o(0) = 0$.

We can use this result to obtain some analytical results for $S_{1;\alpha}(T = 0)$ in the presence of quenched disorder. Since the right-hand side of eq. (5) vanishes for the ordered case, $n_{s;\alpha\alpha}^o(0)q^2/m^* c^2 = (2\pi a_\alpha/\Phi_0)^2 E_\alpha^o(0)/V$, where the superscript refers to the ordered system. Using this equivalence, we may rewrite $S_{1;\alpha}(T)$ as

$$S_{1;\alpha}(T) = \frac{\pi q^2 n_{s;\alpha\alpha}^o(0)}{2m^*}\left(\frac{E_\alpha(T)}{E_\alpha^o(0)} - \frac{n_{s;\alpha\alpha}(T)}{n_{s;\alpha\alpha}^o(0)}\right). \qquad (6)$$

If the disordered system is unfrustrated (all $J_{ij} > 0$), all θ_i are still equal at $T = 0$, and $E_\alpha(T = 0)$ is controlled by the average bond strength $\bar{J}_\alpha$: $E_\alpha(0) = N\bar{J}_\alpha$. In view of this fact, we choose reference system to have $J_\alpha^o = \bar{J}_\alpha$, so that $E_\alpha(0)/E_\alpha^o(0) = 1$.

To further analyze this regime, we use a generalization of a theorem proved by Kirkpatrick[12], which maps the

spin-wave stiffness constant of a random Heisenberg ferromagnet onto the conductance of a disordered conductance network, and generalized to the superfluid density of XY systems (the analog of the spin-wave stiffness constant) in Ref.[13]. The generalized theorem, as applied to the present anisotropic geometry, states that

$$\frac{n_{s;\alpha\alpha}(T=0)}{n_{s;\alpha\alpha}^o(T=0)} = \frac{g_{\alpha\alpha}}{g_{\alpha\alpha}^o}. \qquad (7)$$

Here $g_{\alpha\alpha}$ and $g_{\alpha\alpha}^o$ are components of the conductivity tensor of fictitious random conductance networks in which the bond conductances are J_{ij} and J_{ij}^o, the bond strengths of the real and reference systems.

Using this theorem, we can evaluate $S_{1;\alpha}(0)$ in an isotropic system for weak disorder. We retain our choice of the $\bar{J}_\alpha$ for the reference system. For an isotropic conductance network in d dimensions[14], $g/\bar{g} = 1 - \langle(\delta g)^2\rangle_{dis}/[d\bar{g}^2] + \mathcal{O}((\delta g)^3)$, where $\delta g = g - \bar{g}$ and $\langle...\rangle_{dis}$ denotes an average over configurations of the quenched disorder. Hence, making use of eq. (7), we obtain $n_{s;\alpha\alpha}(0) = n_{s;\alpha\alpha}^o(0)\left(1 - \frac{\langle(J-\bar{J})^2\rangle_{dis}}{\bar{J}^2 d}\right)$, where $d = 2$ or 3 is the dimensionality and $\bar{J}$ is the average bond strength of a bond in this network.

Substituting this expression back into eq. (6) gives our final result for the integrated fluctuation conductivity in the case of weak disorder:

$$\int_0^\infty \sigma_{1;\alpha\alpha}(\omega, T=0)d\omega = \frac{\pi q^2 n_{s;\alpha\alpha}(0)}{2m^*}\frac{\langle(J-\bar{J})^2\rangle_{dis}}{\bar{J}^2 d}. \qquad (8)$$

Here we have used the fact that, $n_{s;\alpha\alpha}(0)$ and $n_{s;\alpha\alpha}^o(0)$ are equal to lowest order in $\langle(J-\bar{J})^2\rangle_{dis}$. The dimensions $d = 2$ or $d = 3$ would be appropriate for the limiting cases of no interplanar coupling (d = 2) or a fully isotropic system (d = 3); presumably a material such as $Bi_2Sr_2Ca_2Cu_2O_{8+\delta}$ would behave similarly to the $d = 2$ case.

The result (8) has several similarities to experiments[6,8]. Most strikingly, at fixed magnitude of the disorder parameter $\langle(J-\bar{J})^2\rangle_{dis}/\bar{J}^2$, $S_{1;\alpha}(T = 0)$ is predicted to scale with the low-temperature superfluid density, in agreement with experiment. Of course, the experiment is carried out at a specific frequency, while the relationship (8) applies to a frequency *integral*, but presumably $\sigma_{1;\alpha\alpha}(\omega)$ behave similarly for any given frequency. Note also that the result does not require the presence or absence of short range order in the bond strength[14], but only that the bond distribution of bonds be macroscopically isotropic in d dimensions.

It is of interest to make a numerical estimate of the integral (8) for parameters appropriate for $Bi_2Sr_2Ca_2Cu_2O_{8+\delta}$. From the London equations, we may write $\pi q^2 n_{s;\alpha\alpha}/(2m^*) = c^2/(8\lambda^2)$. If we take the in-plane penetration depth as $2000\mathring{A}$ and we use a low-temperature conductivity $\sigma_1(\omega, 0)$ of about $10^6\ \Omega^{-1}m^{-1}$

at $\omega \sim 140$ GHz, as suggested in Ref.[8], and we further assume that this value remains roughly constant to low frequencies and cuts off at only slightly higher frequencies, then relation (8) could be satisfied for $\langle (J - \bar{J})^2 \rangle / \bar{J}^2 \sim 10^{-1}$. This is at best an order-of-magnitude estimate, but it suggests at least that the relation (8) is not ruled out by experiment, since mean-square fluctuations in this range seem reasonable.

The present results are entirely independent of the specific dynamics: *any* classical dynamics will give the same value for $S_{1;\alpha}(T)$. Nevertheless, in order to illustrate a possible behavior for $\sigma_1(\omega)$, we have approximated the dynamical response of the phase by the equations describing a coupled array of overdamped Josephson junctions [see, for example, Ref. ([15])]. In this picture, each lattice point is connected by an overdamped Josephson junction which carries three current contributions in parallel: a supercurrent $I_{c;ij} \sin \theta_{ij}$, a normal current through a shunt resistance R_{ij}, and a Langevin noise current $I_{L;ij}(t)$ representing the effects of thermal fluctuations.

We will consider the anisotropic limit, in which the bond strengths J_{ij} vanish for $ij \| c$. In this limit, the supercurrents from bonds in different layers are uncorrelated and eq. (3) reduces to (for conductivity $\sigma_{1;xx}(\omega, T)$ parallel to the a axis)

$$\sigma_{1;xx}(\omega) = \frac{1}{N_s a_z k_B T} \int_0^\infty dt \cos(\omega t) \langle \mathcal{I}_\alpha(0) \mathcal{I}_\alpha(t) \rangle, \quad (9)$$

where N_s is the number of lattice points in one layer, and the sum defining $\mathcal{I}_\alpha$ [eq. (4)] runs over lattice points in a *single* layer. We have evaluated this average by solving the coupled Josephson equations, using the method described in ref.[15].

Figs. 1 and 2 show $\sigma_{1;xx}(\omega, T)$ for this model in two cases. In Fig. 1, all the critical currents and shunt resistance have the same values, I_c and R. In Fig. 2, the $I_{c;ij}$ are uniformly and independently distributed on the interval $(0, 2I_c)$, but all the R's remain identical. In both cases, time is expressed in units of $\hbar/(qRI_c)$, frequency in units of $qRI_c/\hbar$, current in units of I_c, $k_B T$ in units of $\hbar I_c/q$, and therefore $\sigma_{1;xx}(\omega, T)$ in units of $1/(Ra_z)$.

The results for the two cases are strikingly different. For the *ordered* lattice, there is a fluctuation peak in $\sigma_1(\omega, T)$, more prominent at smaller frequencies, centered near the lattice phase-ordering temperature $T_c \sim 0.95\hbar I_c/q^{16}$. [The peak is probably shifted away from this value by finite-size effects in our calculations.] For $T < T_c$, at all values of ω, $\sigma_1(\omega, T)$ falls off sharply towards a very small value at $T = 0$, consistent with the prediction that $S_{1;\alpha\alpha}(0) = 0$. [The decreasing character of $\sigma_1(\omega, T)$ is most evident in the semilog plot.]
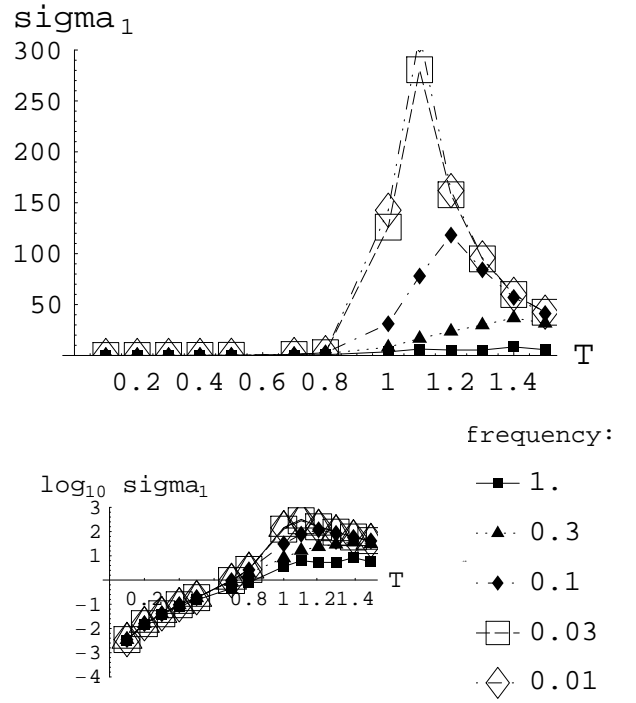

FIG. 1. Fluctuation conductivity $\sigma_1(\omega, T)$, plotted as a function of temperature T for several values of the frequency ω, calculated for an *ordered* 32×32 lattice with periodic boundary conditions. The inset shows same results in a semilog scale, demonstrating that $\sigma_1(\omega, T \to 0) \to 0$. Units are as described in the text, and lines are guides to the eye.

In the *disordered* lattice, $\sigma_1(\omega, T)$ still has a strong fluctuation peak, but in addition has a broad background for all $T < T_c$, which rolls off at larger frequencies. The frequency-dependence is probably Lorentzian, as would be expected if the current-current correlation function in (3) decays exponentially in time. We expect that the relaxation time $\tau(T)$ should be of order $\hbar/(2eRI_c)$. The semilog plot shows clearly that $\sigma_1(\omega, T)$ remains finite even as $T \to 0$.

The disorder-induced broad background in our calculations may appear rather small (much weaker than the peak near T_c). But this background is calculated in units of the strongly temperature-dependent coupling energy J. According to one model, for example[13], $J \propto \Delta^2$, where Δ is the mean-field energy gap, a quantity which decreases to zero at the mean-field transition temperature T_{c0}. When the temperature-dependence of J is properly included, the fluctuation peak may well prove to be much smaller than the low-temperature background, especially when the low-temperature n_s is relatively large; this behavior would be in agreement with experiment[8].
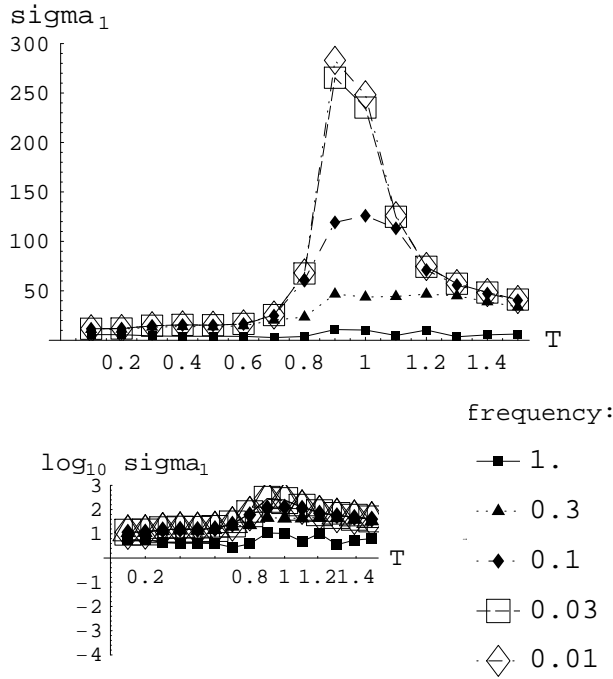
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FIG. 2. $\sigma_1(\omega, T)$ plotted as a function of T for several frequencies in a *disordered* lattice (32×32 array), in which the critical current is uniformly and randomly distributed on the interval $(0, 2I_c)$. The inset shows same results in a semilog scale. Lines are guides for eyes.

We believe that the disorder-induced background conductivity originates as follows. In the absence of disorder, phase fluctuations at low T will occur in the form of long-wavelength, low-frequency "phase phonons." These are loosely analogous to spin waves in ferromagnets, but may be underdamped or overdamped, depending on the dynamics. In the presence of quenched disorder, the density of states of these phase phonons is affected by the scattering of these excitations off the disorder; in addition, the modes of a given frequency, instead of having a unique wave vector, will form a wave packet with a spread in wave vector. Analogous behavior is found in the spin wave spectra of ferromagnets when there is quenched disorder in, e. g., the exchange constants. We speculate that these damped modes produce the extra contribution to the $\sigma_1(\omega, T)$ at low temperatures.

There are several issues which we have not considered in the present work. For example, classical phase fluctuations might be expected to be frozen out by quantum effects at low T, as may happen in low-T_c superconductors[17]. But in some high-T_c materials, the existence of nodal quasiparticles may provide a normal background which would impede this quantum freeze-out. It remains an open question to what temperature classical phase fluctuations persist in such materials as $Bi_2Sr_2Ca_2Cu_2O_{8+\delta}$.

In conclusion, we have demonstrated, both analyti-

cally and numerically, that in a superconductor with *quenched disorder*, the finite-frequency electrical conductivity $\sigma_1(\omega, T)$ remains non-zero at low temperatures if the order parameter has phase fluctuations which can be treated classically. If this assumption is satisfied, this result is quite general, and independent of the dynamical equations obeyed by these fluctuation. For weak disorder, the frequency integral of this fluctuation conductivity scales proportional to the low-temperature superfluid density, in agreement with recent microwave experiments in $Bi_2Sr_2Ca_2Cu_2O_{8+\delta}$.

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